Problem Set 6 due April 15, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider a length 1 vector $\boldsymbol{a} \in \mathbb{R}^n$ (so $||\boldsymbol{a}|| = 1$), and look at the linear transformation:

$$\phi : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{corresponding to the matrix} \qquad A = I - 2\boldsymbol{a}\boldsymbol{a}^T$$

(1) Compute $\boldsymbol{a}^T\boldsymbol{a}$ and show that the matrix A is orthogonal. *(10 points)*

(2) In terms of $\boldsymbol{a}$, what is the subspace of $\mathbb{R}^n$ fixed by ϕ, i.e. the subspace: *(5 points)*

$$\{\boldsymbol{v} \in \mathbb{R}^n \text{ such that } \phi(\boldsymbol{v}) = \boldsymbol{v}\}$$

(3) Compute $\phi(\boldsymbol{a})$ and describe the linear transformation ϕ geometrically (i.e. say what it is called in plain English, and draw a picture in the $n = 3$ case). *(10 points)*

Solution: (1) We have $\boldsymbol{a}^T\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{a} = 1$. So to prove A is orthogonal, we need to show that:

$$A^TA = (I - 2\boldsymbol{a}\boldsymbol{a}^T)^T(I - 2\boldsymbol{a}\boldsymbol{a}^T) = (I - 2aa^T)^2 = I - 4\boldsymbol{a}\boldsymbol{a}^T + 4\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{a}\boldsymbol{a}^T = I - 4\boldsymbol{a}\boldsymbol{a}^T + 4\boldsymbol{a}\boldsymbol{a}^T = I$$

as required.

Grading Rubric: 5 points for $\boldsymbol{a}^T\boldsymbol{a} = 1$ and 5 points for correct justification of orthogonality.

(2) The subspace we need to compute is $N(A - I) = N(-2\boldsymbol{a}\boldsymbol{a}^T)$. Explicitly:

$$\boldsymbol{v} \in N(-2\boldsymbol{a}\boldsymbol{a}^T) \quad \Leftrightarrow \quad -2\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v} = 0 \quad \Leftrightarrow \quad \boldsymbol{a}^T\boldsymbol{v} = 0 \quad \Leftrightarrow \quad \boldsymbol{a} \perp \boldsymbol{v}$$

so the subspace in question is the orthogonal complement of $\boldsymbol{a}$.

Grading Rubric

- Correct justification as to why the subspace is the orthogonal complement of $\boldsymbol{a}$ *(5 points)*
- Said that the subspace is $N(A - I)$ or $N(-2\boldsymbol{a}\boldsymbol{a}^T)$ *(2 points)*
- No answer, or inadequate justification *(0 points)*

(3) We have:

$$\phi(\boldsymbol{a}) = (I - 2\boldsymbol{a}\boldsymbol{a}^T)\boldsymbol{a} = \boldsymbol{a} - 2\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{a} = \boldsymbol{a} - 2\boldsymbol{a} = -\boldsymbol{a}$$

Since ϕ flips the sign of $\boldsymbol{a}$, and preserves any vector perpendicular to $\boldsymbol{a}$, we conclude that ϕ is reflection in the plane perpendicular to $\boldsymbol{a}$ (if you don't like the word "plane" here because we're in n–dimensional space, think "orthogonal complement").

Grading Rubric 4 points for the computation of $\phi(\boldsymbol{a})$, 3 points for identifying the linear transformation as a reflection in the plane perpendicular to $\boldsymbol{a}$, and 3 points for a representative picture.

Problem 2:

Consider the function:

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y + 2 \\ 3x - y - 1 \end{bmatrix}$$

(1) Explain why f is <u>not</u> a linear transformation. *(5 points)*

(2) Find a linear transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and a translation $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$f = \phi \circ g$$

(a translation is a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $g(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{a}$ for a fixed vector $\boldsymbol{a}$). *(10 points)*

Solution: (1) Note that $f(0) \neq 0$, so f cannot be a linear transformation. Alternatively, one could show that:

$$f\left(\begin{bmatrix} x + x' \\ y + y' \end{bmatrix}\right) \neq f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + f\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) \tag{1}$$

through an example or conceptual argument, or that:

$$f\left(\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}\right) \neq \lambda f\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) \tag{2}$$

through an example or conceptual argument.

Grading Rubric 5 points for any of the arguments above. -2 points if the student only states properties (1) or (2) without a reason or without a counterexample.

(2) If it weren't for the $+2$ and -1 in the top and bottom entries of f, then it would actually be a linear transformation. Therefore, let:

$$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ 3x - y \end{bmatrix}$$

and let us seek to define a translation:

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

for some to-be-determined numbers a and b. We need $f = \phi \circ g$, i.e.:

$$\begin{bmatrix} 2x - y + 2 \\ 3x - y - 1 \end{bmatrix} = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \phi\left(g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \phi\left(\begin{bmatrix} x + a \\ y + b \end{bmatrix}\right) = \begin{bmatrix} 2x - y + 2a - b \\ 3x - y + 3a - b \end{bmatrix}$$

In order for the formula above to hold for all x, y, we need:

$$\begin{cases} 2 = 2a - b \\ -1 = 3a - b \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

You can solve the system above for a, b in many ways, but it's easy to explicitly multiply with the inverse matrix, and get:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$$

So we need $a = -3$ and $b = -8$, and this determines the translation g.

Grading Rubric

- Correct guess and explicit check that $f = \phi \circ g$ holds *(10 points)*
- Correct guess but missing justification or check *(6 points)*
- Identified the correct ϕ, but has the wrong g (minor typos allowed) *(3 points)*
- Significantly incorrect answer. *(0 points)*

Problem 3:

Consider the linear transformation:

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad f(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{where} \qquad A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

(1) Find a basis $\boldsymbol{w}_1, \boldsymbol{w}_2$ of $\mathbb{R}^2$ such that $f(\boldsymbol{e}_1) = \boldsymbol{w}_1$ and $f(\boldsymbol{e}_2) = \boldsymbol{w}_2$, where $\boldsymbol{e}_i$ is the i–th coordinate unit vector. Compute the matrix B which represents f in the new basis, i.e.:

$$f(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) = (b_{11}x_1 + b_{12}x_2 + b_{13}x_3)\boldsymbol{w}_1 + (b_{21}x_1 + b_{22}x_2 + b_{23}x_3)\boldsymbol{w}_2$$

and say explicitly how B relates to A (*Hint: B should be equal to a matrix times A*). *(10 points)*

(2) Construct a basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of $\mathbb{R}^3$ such that $f(\boldsymbol{v}_1) = \boldsymbol{e}_1$, $f(\boldsymbol{v}_2) = \boldsymbol{e}_2$, $f(\boldsymbol{v}_3) = 0$, where $\boldsymbol{e}_i$ is the i–th coordinate unit vector. Compute the matrix C which represents f in the new basis, i.e.:

$$f(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3) = (c_{11}x_1 + c_{12}x_2 + c_{13}x_3)\boldsymbol{e}_1 + (c_{21}x_1 + c_{22}x_2 + c_{23}x_3)\boldsymbol{e}_2$$

and say explicitly how C relates to A (*Hint: C should be equal to A times a matrix*). *(10 points)*

Solution: (1) One can explicitly compute:

$$\boldsymbol{w}_1 = f(\boldsymbol{e}_1) = A\boldsymbol{e}_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{w}_2 = f(\boldsymbol{e}_2) = A\boldsymbol{e}_2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Explicitly, we have:

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = f(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3)$$

and:

$$\left[\ \boldsymbol{w}_1 \mid \boldsymbol{w}_2\ \right] B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (b_{11}x_1 + b_{12}x_2 + b_{13}x_3)\boldsymbol{w}_1 + (b_{21}x_1 + b_{22}x_2 + b_{23}x_3)\boldsymbol{w}_2$$

We are required to define B so that the right-hand sides of the formulas above are equal for all x_1, x_2, x_3, so the only thing we can do is to ensure that the left-hand sides are equal, i.e.:

$$A = \left[\ \boldsymbol{w}_1 \mid \boldsymbol{w}_2\ \right] B \quad \Rightarrow \quad B = \left[\ \boldsymbol{w}_1 \mid \boldsymbol{w}_2\ \right]^{-1} A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Grading Rubric: 4 points for $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, and 6 points for B (only 3 of the latter 6 points if B is given without any justification, e.g. the derivation above).

(2) Let $\boldsymbol{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and let us solve for:

$$f(\boldsymbol{v}_1) = \boldsymbol{e}_1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

One can do row operations to put the matrix above in reduced row echelon form (namely subtract row 1 from row 2, divide row 2 by 3, and then add row 2 to row 1) and the system above is equivalent to:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Here z is the free variable and x and y are the pivot variables, so we can get a solution by setting $z = 0$ and solving for x and y:

$$x = \frac{2}{3} \quad \text{and} \quad y = -\frac{1}{3} \qquad \Rightarrow \qquad \boldsymbol{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

By a similar reasoning, one gets:

$$\boldsymbol{v}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Explicitly, we have:

$$A \left[\ \boldsymbol{v}_1 \ \middle| \ \boldsymbol{v}_2 \ \middle| \ \boldsymbol{v}_3 \ \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = f(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3)$$

and:

$$C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (c_{11}x_1 + c_{12}x_2 + c_{13}x_3)\boldsymbol{e}_1 + (c_{21}x_1 + c_{22}x_2 + c_{23}x_3)\boldsymbol{e}_2$$

We are required to define C so that the right-hand sides of the formulas above are equal for all x_1, x_2, x_3, so the only thing we can do is to ensure that the left-hand sides are equal, i.e.:

$$C = A \left[\ \boldsymbol{v}_1 \ \middle| \ \boldsymbol{v}_2 \ \middle| \ \boldsymbol{v}_3 \ \right] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -1 \\ -\frac{1}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Grading Rubric: 6 points for $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, and 4 points for C (only 2 of the latter 4 points if C is given without any justification, e.g. the derivation above).

Problem 4:

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

(1) Compute $\det A$ by row operations (i.e. $\pm$ product of pivots). *(10 points)*

(2) Compute $\det A$ by cofactor expansion. *(10 points)*

Note: you may use an explicit formula for 2×2 determinants, but not for bigger ones.

Solution: (1) Let's put A in row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -10 & -13 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{r_3 - 3r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -10 & -13 \\ 0 & -2 & -8 & -10 \\ 2 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{r_4 - 2r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -10 & -13 \\ 0 & -2 & -8 & -10 \\ 0 & -1 & -2 & -7 \end{bmatrix}$$

$$\xrightarrow{r_3 - \frac{2}{7}r_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -10 & -13 \\ 0 & 0 & -\frac{36}{7} & -\frac{44}{7} \\ 0 & -1 & -2 & -7 \end{bmatrix} \xrightarrow{r_4 - \frac{1}{7}r_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -10 & -13 \\ 0 & 0 & -\frac{36}{7} & -\frac{44}{7} \\ 0 & 0 & -\frac{4}{7} & -\frac{36}{7} \end{bmatrix} \xrightarrow{r_4 - \frac{1}{9}r_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -10 & -13 \\ 0 & 0 & -\frac{36}{7} & -\frac{44}{7} \\ 0 & 0 & 0 & -\frac{40}{9} \end{bmatrix}$$

Since we have not done any row exchanges, the determinant is equal to the product of the pivots, which is -160.

Grading Rubric

- Correct row reduction and determinant *(10 points)*
- Correct row reduction but incorrect determinant *(7 points)*
- Used a different method to get determinant *(4 points)*
- Significantly wrong method *(0 points)*

(2) Let's do cofactor expansion along the first row:

$$\det A = 1 \cdot \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 4 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 4 & 1 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 4 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix} - 4 \cdot \det \begin{bmatrix} 4 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} \tag{3}$$

To shake things up, let's compute these four 3×3 determinant by cofactor expansion along the <u>second</u> column:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix} = -2 \cdot \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = -2 \cdot (-2) + 1 \cdot (-8) - 4 \cdot (-10) = 36$$

$$\det \begin{bmatrix} 4 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 4 & 1 \end{bmatrix} = -2 \cdot \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = -2 \cdot (-1) + 1 \cdot (-2) - 4 \cdot (-1) = 4$$

$$\det \begin{bmatrix} 4 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix} = -1 \cdot \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} - 3 \cdot \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = -1 \cdot (-1) + 4 \cdot (-2) - 3 \cdot (-1) = -4$$

$$\det \begin{bmatrix} 4 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} = -1 \cdot \det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - 3 \cdot \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -1 \cdot 10 + 4 \cdot 12 - 3 \cdot (-2) = 44$$

Plugging this into the formula (3) yields $\det A = 36 - 8 - 12 - 176 = -160$.

Grading Rubric

- Correct cofactor expansion *(10 points)*
- -2 points for every wrong sign (no more than -4 points deducted this way)
- -1 points for every algebra error (no more than -2 points deducted this way)
- Significantly wrong method *(0 points)*

Problem 5:

Consider the matrix:

$$A_n = \begin{bmatrix} 0 & x_1 & x_2 & \dots & x_n \\ x_1 & 1 & 0 & \dots & 0 \\ x_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \dots & 1 \end{bmatrix}$$

Compute a recursive formula for $\det A_n$, and then obtain an explicit formula. *(20 points)*

Solution: Let's do cofactor expansion along the last (i.e. the $n+1$) row:

$$\det A_n = (-1)^{n+1+1} x_n \cdot \det \begin{bmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 0 & x_1 & x_2 & \dots & x_{n-1} \\ x_1 & 1 & 0 & \dots & 0 \\ x_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & 0 & 0 & \dots & 1 \end{bmatrix}$$

Note that the second determinant is just $\det A_{n-1}$. Meanwhile, we can compute the first determinant by cofactor expansion along the last column:

$$\det A_n = (-1)^{n+1+1} x_n \cdot (-1)^{n+1} x_n \cdot \det \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} + \det A_{n-1} = -x_n^2 + \det A_{n-1}$$

We can iterate the formula above (and the obvious fact that $\det A_0 = 0$), and get:

$$\det A_n = -x_1^2 - ... - x_n^2$$

Grading Rubric

- Correct proof by cofactor expansion *(20 points)*
- Correct proof by cofactor expansion, but minor computation errors *(16-18 points)*
- Correct proof in a particular ($n \geq 2$) case *(10 points)*
- Attempted different approach to compute determinants, but no progress *(5 points)*
- No solution *(0 points)*